Automatic Control (1)

By

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Lecture (3) – Part (2)

Block Diagram Reduction using Decomposition

&

State-Space Representation

Decomposition

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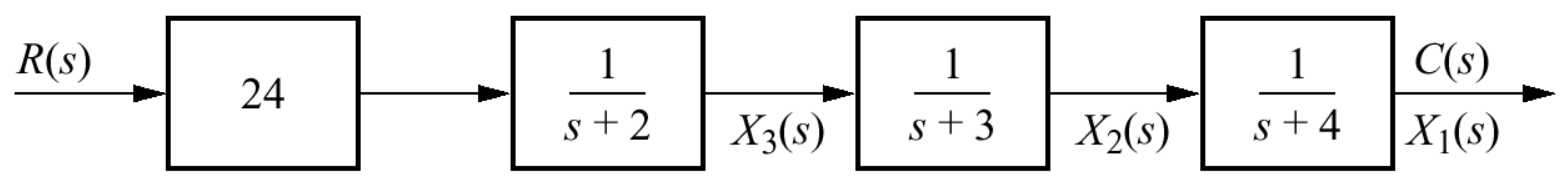
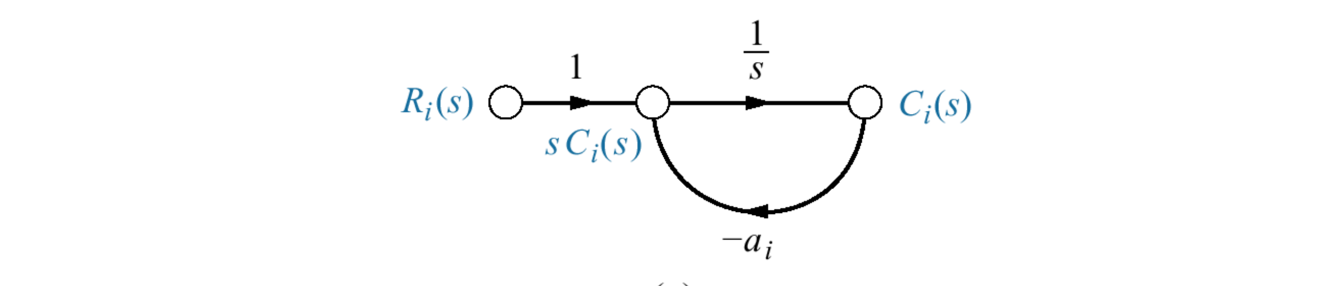
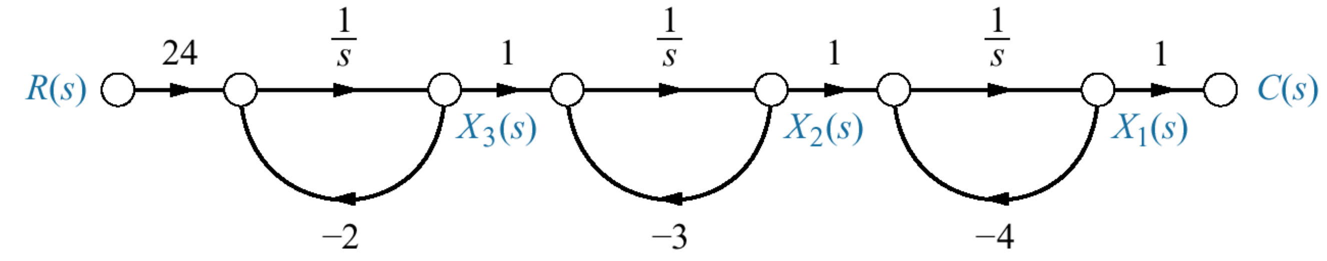
Alternate Representation: Cascade Form



$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)}$$



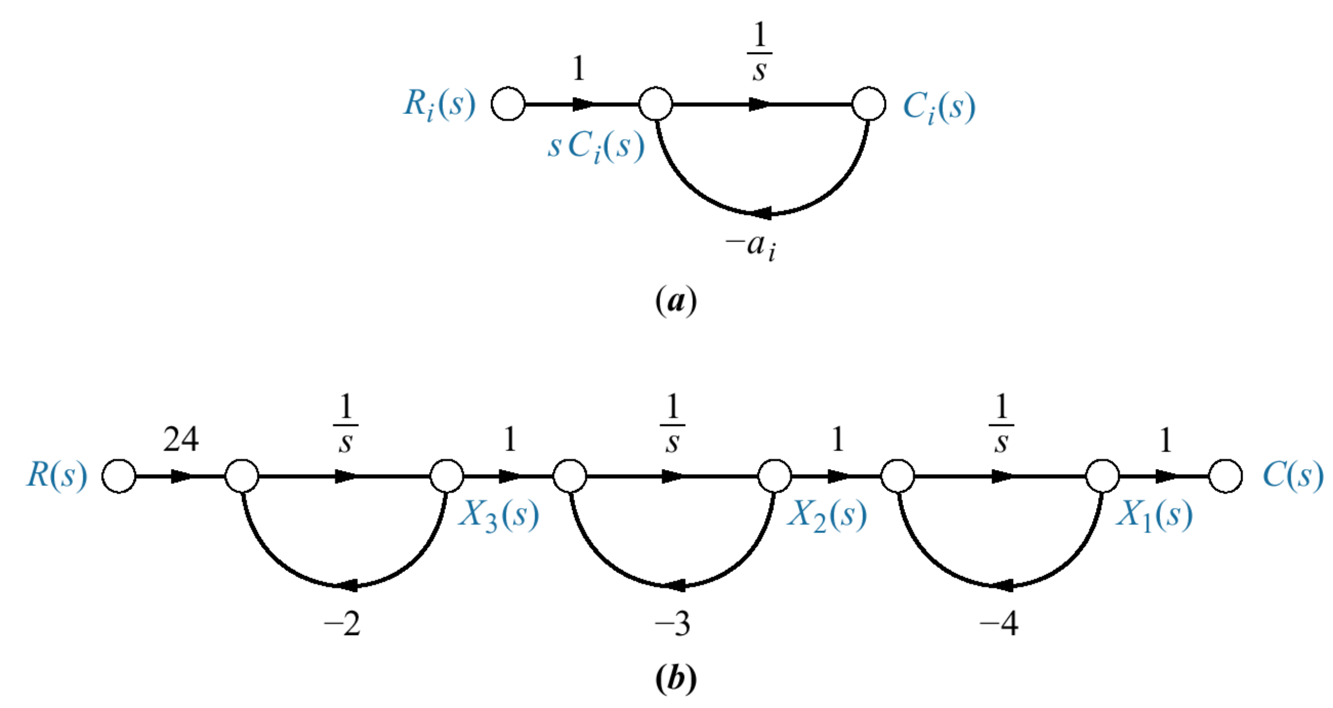
(a)



(b)

Ebrahim

Alternate Representation: Cascade Form



(a)

(b)

$$
\begin{aligned}
\dot{x}_1 &= -4x_1 + x_2 \\
\dot{x}_2 &= \quad -3x_2 + x_3 \\
\dot{x}_3 &= \quad\quad -2x_3 + 24r \\
y &= c(t) = x_1
\end{aligned}
$$

$$
\dot{X} = \begin{bmatrix} -4 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 24 \end{bmatrix} r
$$

$$
y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X
$$

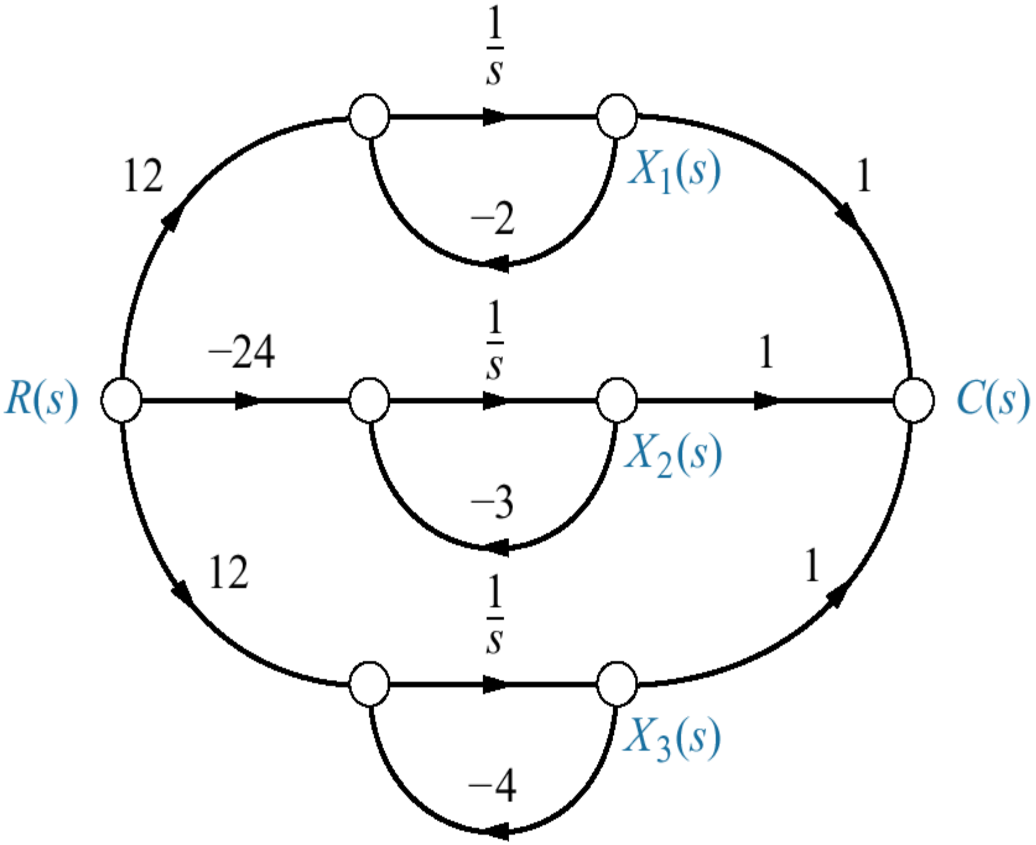
Alternate Representation: Parallel Form

$$\frac{C(s)}{R(s)} = \frac{24}{(s+2)(s+3)(s+4)} = \frac{12}{(s+2)} - \frac{24}{(s+3)} + \frac{12}{(s+4)}$$

$$\begin{aligned}
\dot{x}_1 &= -2x_1 && && +12r \\
\dot{x}_2 &= && -3x_2 && -24r \\
\dot{x}_3 &= && -4x_3 && +12r \\
y &= c(t) = x_1 + x_2 + x_3
\end{aligned}$$

$$\dot{X} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix} X + \begin{bmatrix} 12 \\ -24 \\ 12 \end{bmatrix} r$$

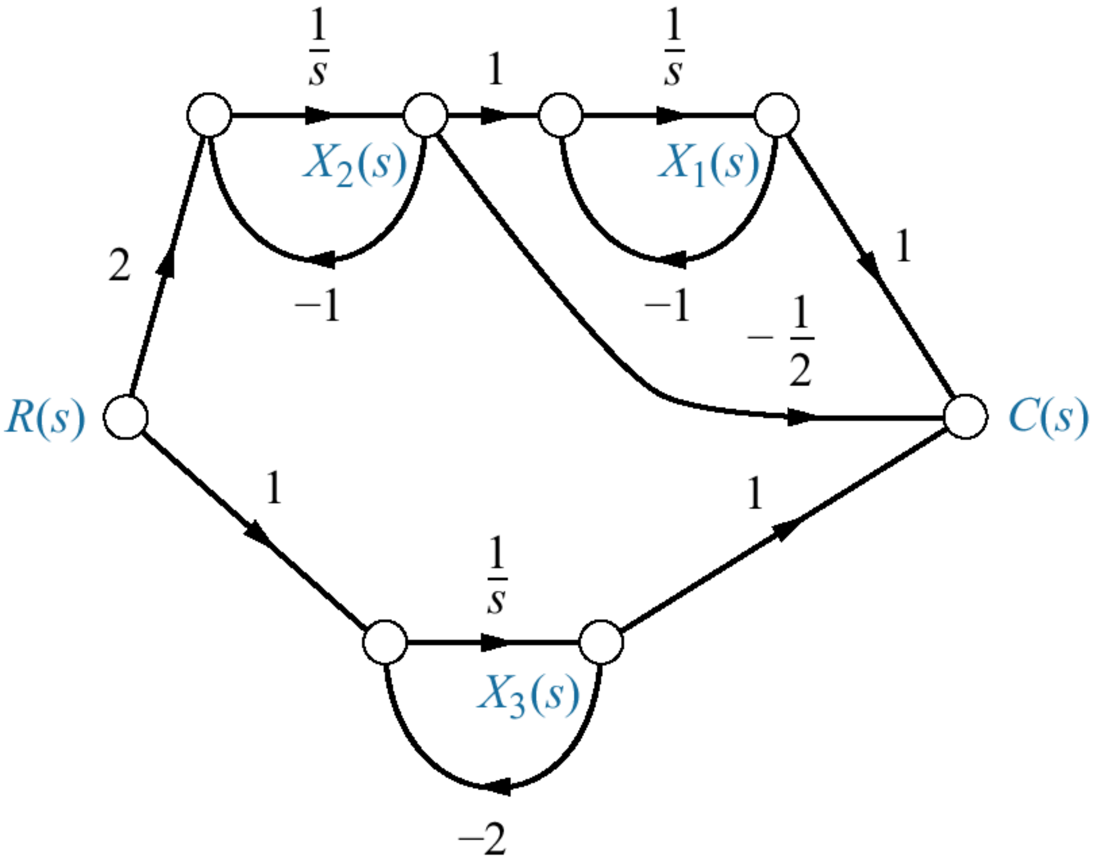
$$y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} X$$



Alternate Representation: Parallel Form Repeated roots

$$\frac{C(s)}{R(s)} = \frac{(s+3)}{(s+1)^2(s+2)} = \frac{2}{(s+1)^2} - \frac{1}{(s+1)} + \frac{1}{(s+2)}$$

$$\dot{x}_1 = -x_1 + x_2$$
$$\dot{x}_2 = x_2 + 2r$$
$$\dot{x}_3 = -2x_3 + r$$
$$y = c(t) = x_1 - 1/2x_2 + x_3$$



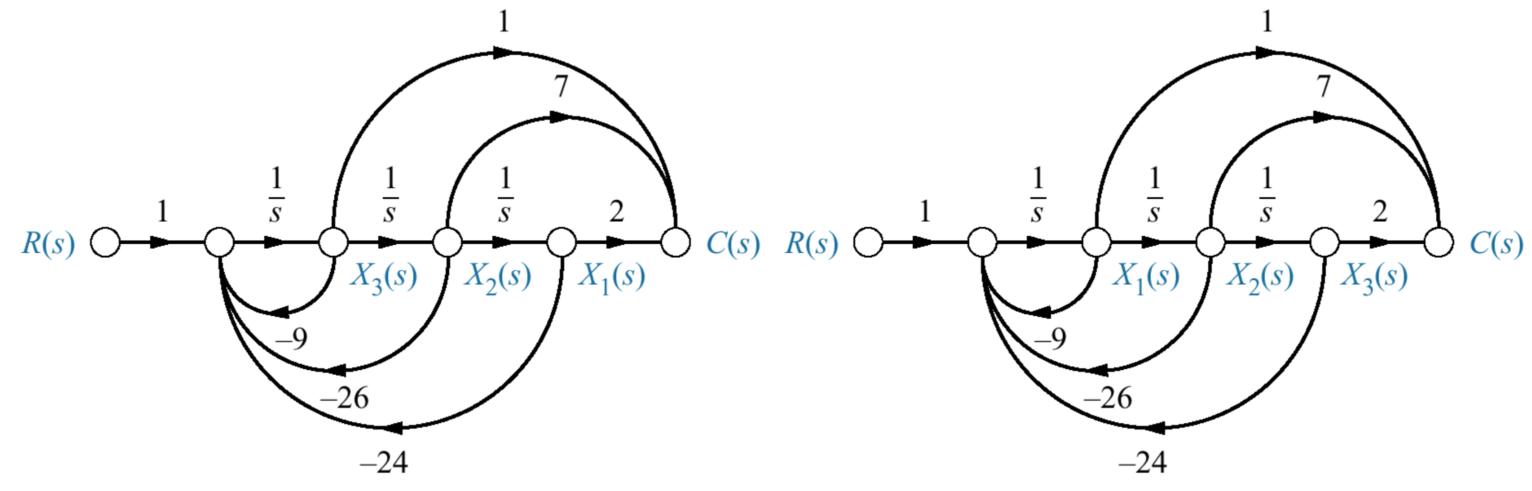
$$\dot{X} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} r$$

$$y = \begin{bmatrix} 1 & -1/2 & 1 \end{bmatrix} X$$

Alternate Representation: controller canonical form

G(s) = C(s)/R(s) = $(s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

This form is obtained from the phase-variable form simply by ordering the phase variable in reverse order



(a)

(b)

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} r$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -9 & -26 & -24 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 7 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Alternate Representation: controller canonical form

System matrices that contain the coefficients of the characteristic polynomial are called *companion matrices* to the characteristic polynomial.

Phase-variable form result in lower companion matrix

Controller canonical form results in upper companion matrix

Alternate Representation: observer canonical form

Observer canonical form so named for its use in the design of observers

$G(s) = C(s)/R(s) = (s^2 + 7s + 2)/(s^3 + 9s^2 + 26s + 24)$

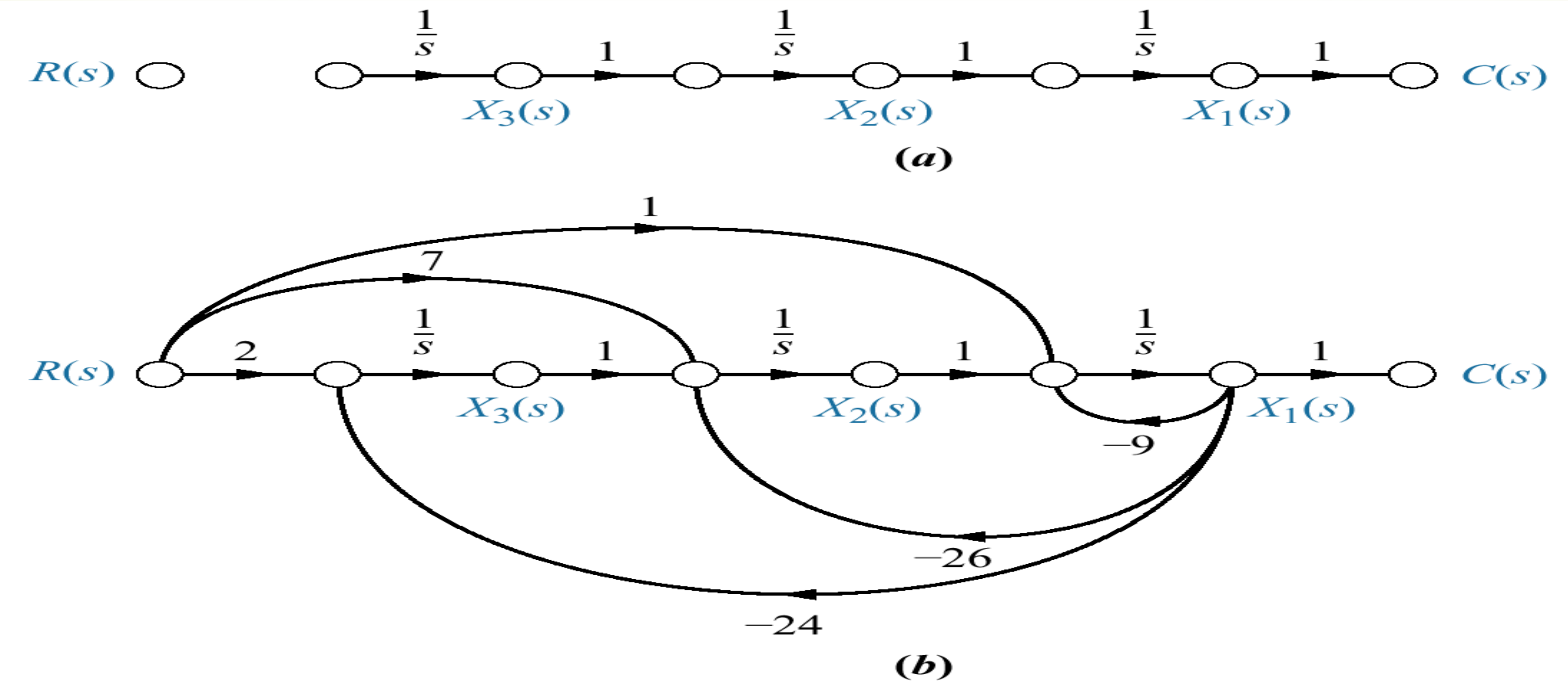
$= (1/s+7/s^2 +2/s^3)/(1+9/s+26/s^2 +24/s^3)$

Cross multiplying

$(1/s+7/s^2 +2/s^3)R(s) = (1+9/s+26/s^2 +24/s^3) C(s)$

And $C(s) = 1/s[R(s)-9C(s)] +1/s^2[7R(s)-26C(s)]+1/s^3[2R(s)-24C(s)]$

$= 1/s\{ [R(s)-9C(s)] + 1/s \{[7R(s)-26C(s)]+1/s [2R(s)-24C(s)]\}\}$



(a)

(b)

Alternate Representation: observer canonical form

$$\dot{x}_1 = -9x_1 + x_2 \qquad + r$$

$$\dot{x}_2 = -26x_1 \qquad + x_3 + 7r$$

$$\dot{x}_3 = -24x_1 \qquad + 2r$$

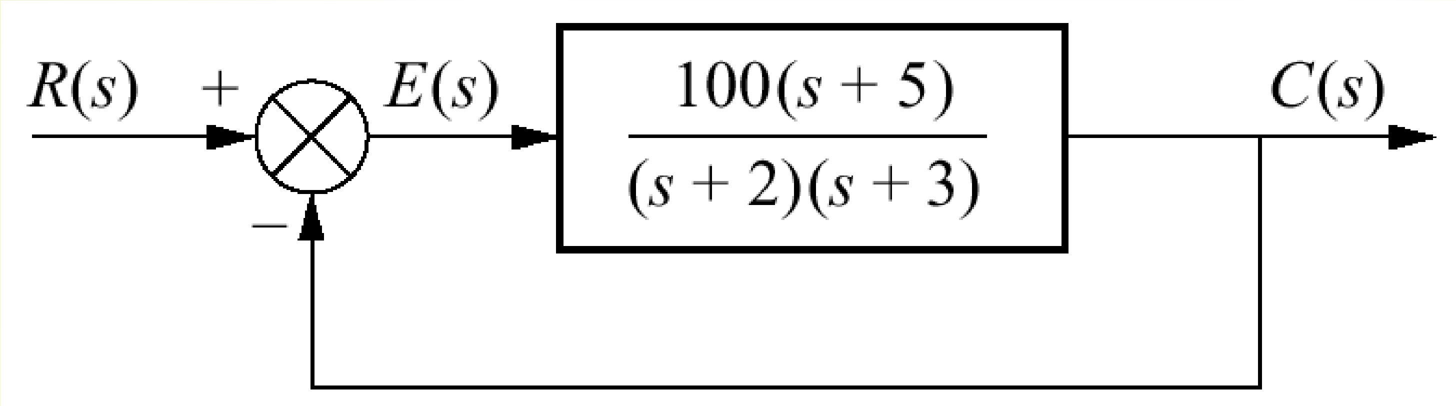
$$y = c(t) = x_1$$

$$\dot{X} = \begin{bmatrix} -9 & 1 & 0 \\ -26 & 0 & 1 \\ -24 & 0 & 0 \end{bmatrix} X + \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} r$$

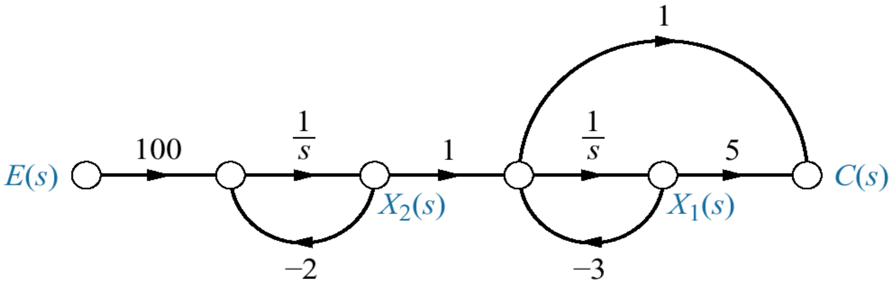
$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} X$$

Note that the observer form has A matrix that is transpose of the controller canonical form, B vector is the transpose of the controller C vector, and C vector is the transpose of the controller B vector. The 2 forms are called duals.

Feedback Control System Example

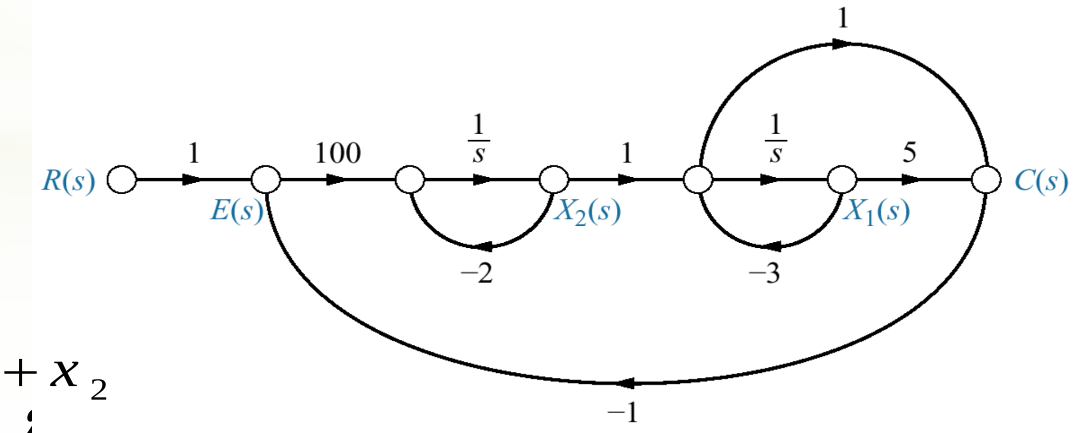


Problem Represent the feedback control system shown in state space. Model the forward transfer function in cascade form.



(*a*)

Solution first we model the forward transfer function as in (a), Second we add the feedback and input paths as shown in (b) complete system. Write state equations



(*b*)

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = -2x_2 + 100(r - c)$$

$$\text{but } c = 5x_1 + (x_2 - 3x_1) = 2x_1 + x_2$$

Feedback Control System Example

$$\dot{x}_1 = -3x_1 + x_2$$

$$\dot{x}_2 = -200x_1 - 102x_2 + 100r$$

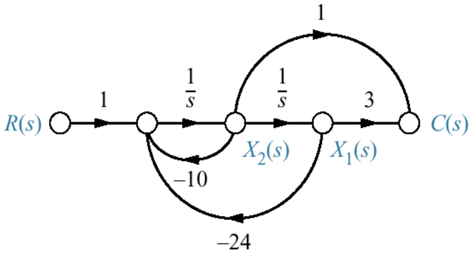
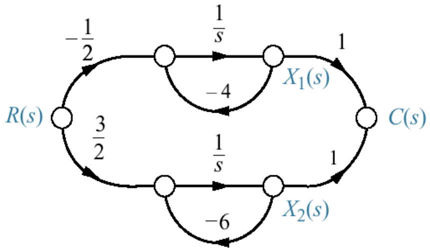
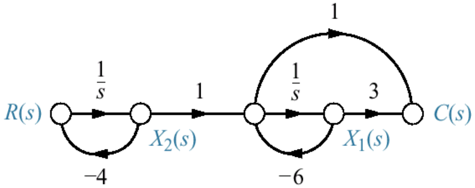
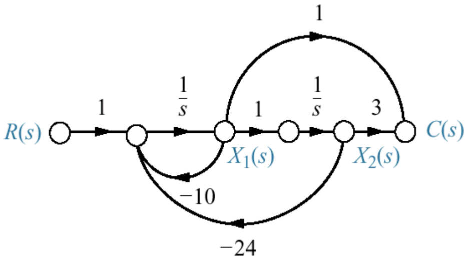
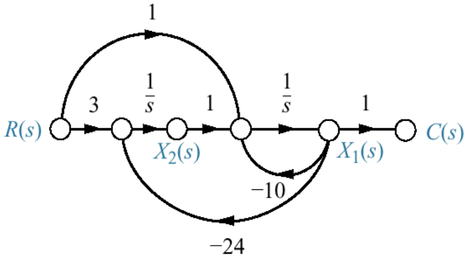
$$y = c(t) = 2x_1 + x_2$$

$$\dot{X} = \begin{bmatrix} -3 & 1 \\ -200 & -102 \end{bmatrix} X + \begin{bmatrix} 0 \\ 100 \end{bmatrix} r$$

$$y = \begin{bmatrix} 2 & 1 \end{bmatrix} X$$

State-space forms for

C(s)/R(s) =(s+ 3)/[(s+ 4)(s+ 6)].
Note: y = c(t)

Form	Transfer Function	Signal-Flow Diagram	State Equations
Phase variable	$\frac{1}{(s^2+10s+24)} * (s+3)$	 1 1/s 1/s 3 1 R(s) C(s) $X_2(s)$ $X_1(s)$ −10 −24	$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -24 & -10 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [3 \quad 1]\mathbf{x}$
Parallel	$\frac{-1/2}{(s+4)} + \frac{3/2}{s+6}$	 $-\frac{1}{2}$ 1/s 1 $X_1(s)$ −4 R(s) C(s) $\frac{3}{2}$ 1/s 1 $X_2(s)$ −6	$\dot{\mathbf{x}} = \begin{bmatrix} -4 & 0 \\ 0 & -6 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -\frac{1}{2} \\ \frac{3}{2} \end{bmatrix} r$ $y = [1 \quad 1]\mathbf{x}$
Cascade	$\frac{1}{(s+4)} * \frac{(s+3)}{(s+6)}$	 1/s 1 1 1/s 3 R(s) C(s) $X_2(s)$ $X_1(s)$ −4 −6	$\dot{\mathbf{x}} = \begin{bmatrix} -6 & 1 \\ 0 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$ $y = [-3 \quad 1]\mathbf{x}$
Controller canonical	$\frac{1}{(s^2+10s+24)} * (s+3)$	 1 1 1/s 1 1/s 3 R(s) C(s) $X_1(s)$ $X_2(s)$ −10 −24	$\dot{\mathbf{x}} = \begin{bmatrix} -10 & -24 \\ 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} r$ $y = [1 \quad 3]\mathbf{x}$
Observer canonical	$\frac{\frac{1}{s}+\frac{3}{s^2}}{1+\frac{10}{s}+\frac{24}{s^2}}$	 1 3 1/s 1 1/s 1 R(s) C(s) $X_2(s)$ $X_1(s)$ −10 −24	$\dot{\mathbf{x}} = \begin{bmatrix} -10 & 1 \\ -24 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 \\ 3 \end{bmatrix} r$ $y = [1 \quad 0]\mathbf{x}$

State-Space Representation

Mathematical Models

1- *Classical* or *frequency-domain* technique

2- *State-Space* or *Modern* or *Time-Domain* technique

Classical or Frequency-Domain Technique

- Advantages

  - Converts differential equation into algebraic equation via transfer functions.

  - Rapidly provides stability & transient response info.

- Disadvantages

  - Applicable only to Linear, Time-Invariant (LTI) systems or their close approximations.

LTI limitation became a problem circa 1960 when space applications became important.

State-Space or Modern or Time-Domain Technique

- Advantages

  - Provides a unified method for modeling, analyzing, and designing a wide range of systems using matrix algebra.

  - Nonlinear, Time-Varying, Multivariable systems

- Disadvantages

  - Not as intuitive as classical method.

  - Calculations required before physical interpretation is apparent

State-Space Representation

An LTI system is represented in state-space format by the vector-matrix differential equation (DE) as:

$$\underline{\dot{x}}(t) = A\underline{x}(t) + B\underline{u}(t)$$ Dynamic equation (s)

$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t)$$ Measurement equations

$$\textit{with } t \geq t_0 \textit{ and initial conditions } \underline{x}(t_0).$$

The vectors **x, y,** and **u** are the *state*, *output* and *input vectors.*

The matrices A, B, C, and D are the *system*, *input*, *output*, and *feedforward matrices*.

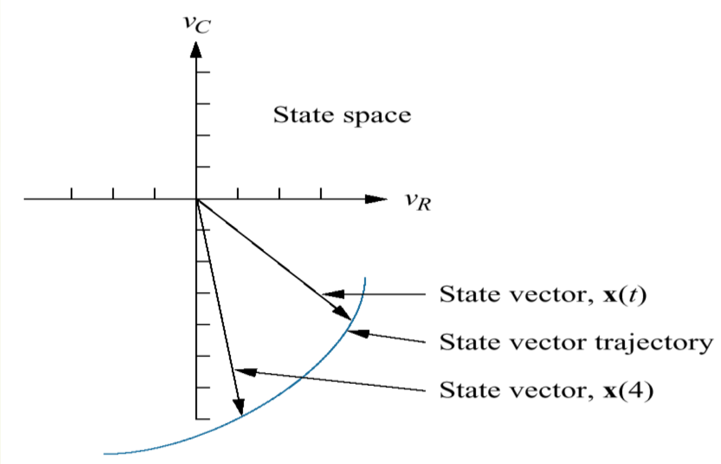
Definitions

- *System variables*: Any variable that responds to an input or initial conditions.
- *State variables*: The smallest set of *linearly independent* system variables such that the initial condition set and applied inputs *completely determine the future behavior* of the set.

Linear Independence: A set of variables is linearly independent if none of the variables can be written as a linear combination of the others.

Definitions

- *State vector:* An (n x 1) column vector whose elements are the state variables.
- *State space:* The n-dimensional space whose axes are the state variables.



Graphic representation
of state space
and a state vector

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The minimum number of state variables is equal to:

- The order of the DE’s describing the system.
- The order of the denominator polynomial of its transfer function model.
- The number of independent energy storage elements in the system.

Remember the state variables must be linearly independent! If not, you may not be able to solve for all the other system variables, or even write the state equations.

Dynamic equation

$$\frac{d}{dt}x(t) = Ax(t) + Bu(t)$$

State equation

$$y(t) = Cx(t) + Du(t)$$

Output equation

State variable

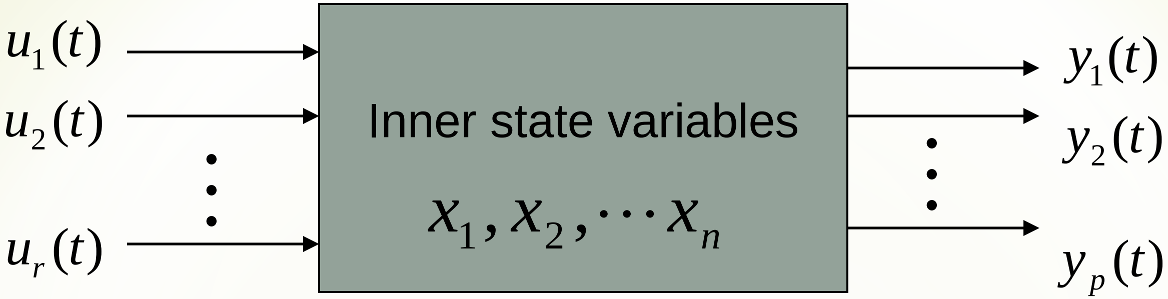
$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} \qquad u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_r(t) \end{bmatrix}_{r\times 1} \qquad y(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_p(t) \end{bmatrix}_{p\times 1} \qquad x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{bmatrix}_{n\times 1}$$

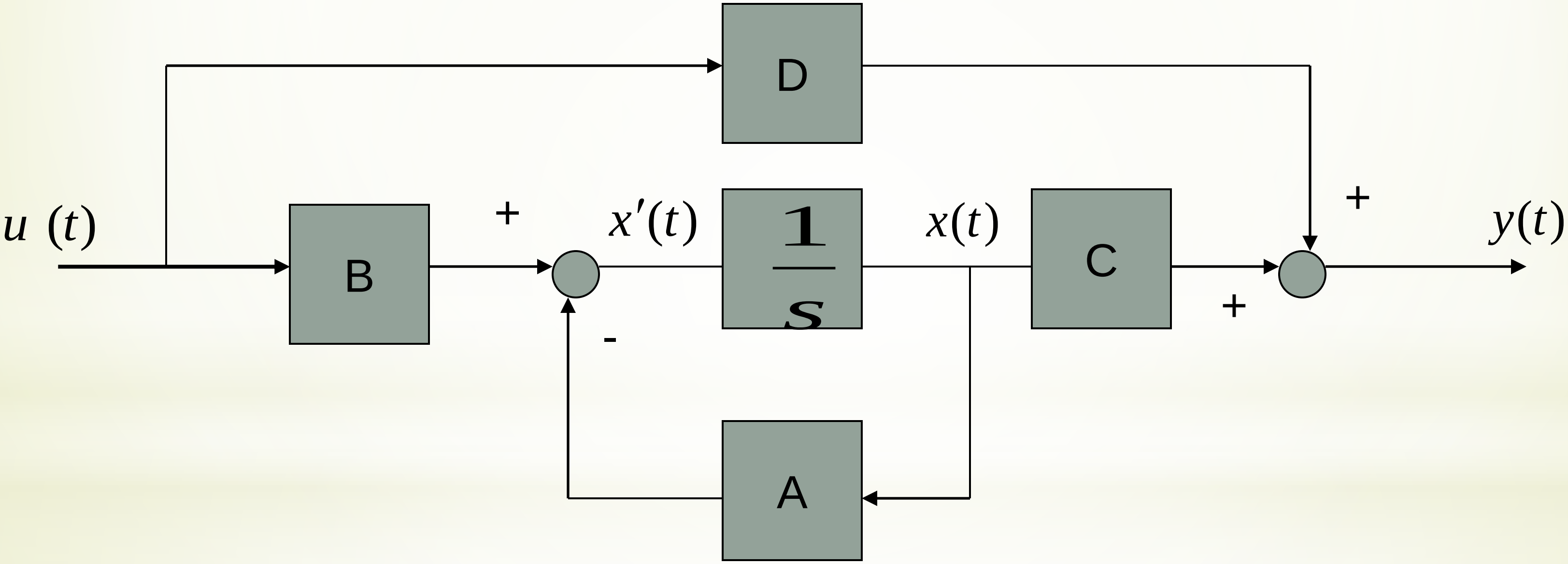
State space

r- input

p- output

$$A = \begin{bmatrix} \quad n \times n \quad \end{bmatrix} \qquad B = \begin{bmatrix} \quad n \times r \quad \end{bmatrix} \qquad C = \begin{bmatrix} \quad p \times n \quad \end{bmatrix} \qquad D = \begin{bmatrix} \quad p \times r \quad \end{bmatrix}$$


$u_1(t)$
$u_2(t)$
$u_r(t)$
Inner state variables
$x_1, x_2, \cdots x_n$
$y_1(t)$
$y_2(t)$
$y_p(t)$


D
$u\ (t)$
B
+
$x'(t)$
$\frac{1}{s}$
$x(t)$
C
+
$y(t)$
-
+
A

Dynamical equation → Transfer function

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

Laplace transform

$$sX(s) - x(0) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

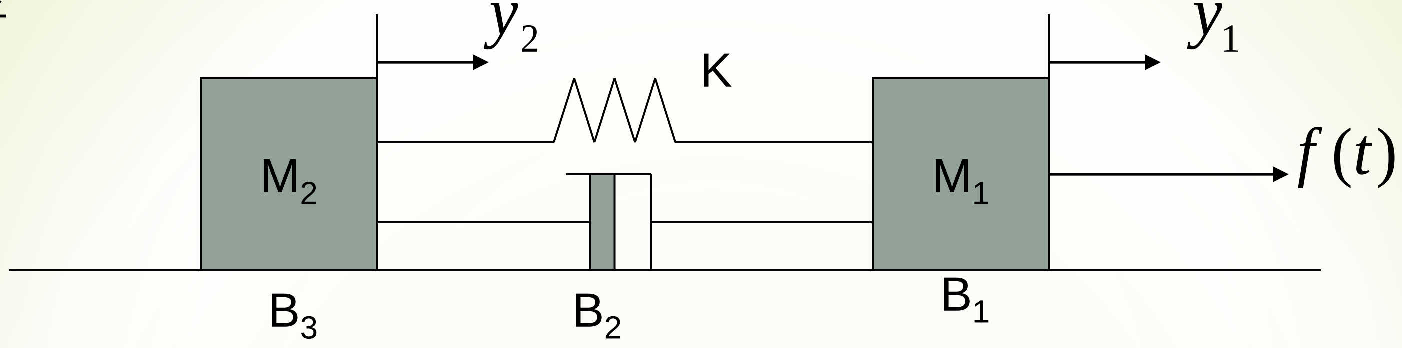
$$assume \quad x(0) = 0$$

$$X(s) = (sI - A)^{-1}BU(s)$$
$$Y(s) = [C(sI - A)^{-1}B + D]U(s)$$

matrix

Transfer function

Example

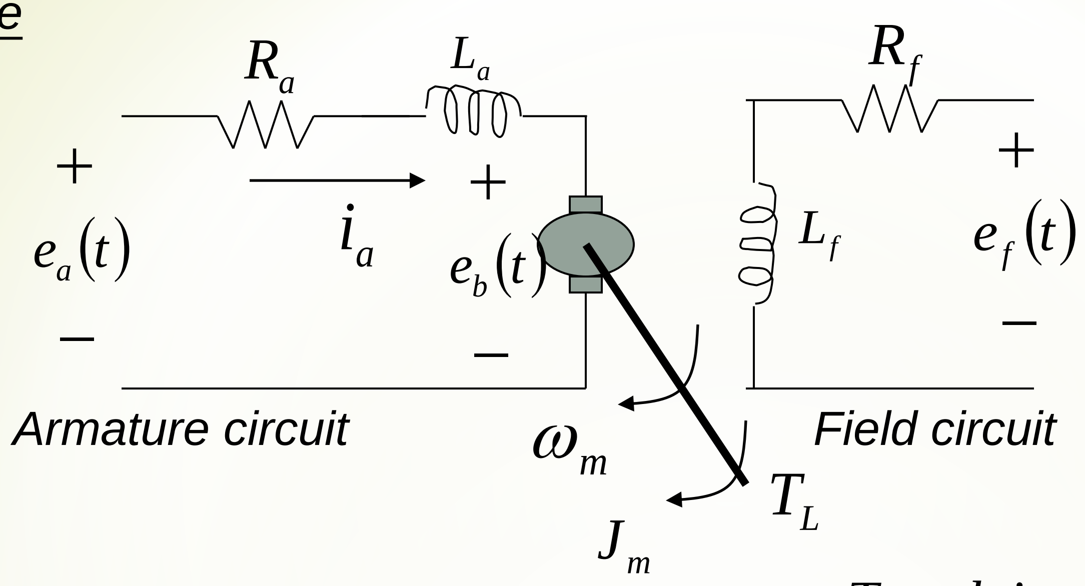


$$M_1\ddot{y}_1 + B_1\dot{y}_1 + B_2(\dot{y}_1 - \dot{y}_2) + K(y_1 - y_2) = f(t)$$

$$M_2\ddot{y}_2 + B_3\dot{y}_2 + B_2(\dot{y}_2 - \dot{y}_1) + K(y_2 - y_1) = 0$$

$$let \begin{cases} x_1 = y_1 - y_2 \\ x_2 = \dot{y}_1 \\ x_3 = \dot{y}_2 \end{cases} \Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -\dfrac{K}{M_1} & -\dfrac{(B_1 + B_2)}{M_1} & \dfrac{B_2}{M_1} \\ \dfrac{K}{M_2} & \dfrac{B_2}{M_2} & -\dfrac{(B_2 + B_3)}{M_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M_1} \\ 0 \end{bmatrix} f(t)$$

Example



$$\begin{cases} e_a(t) = R_a i_a + e_b + L_a \dfrac{di_a}{dt} \\ e_b = k_b \omega_m \end{cases}$$

$$\frac{di_a}{dt} = \frac{-R_a}{L_a} i_a - \frac{k_b}{L_a}\omega_m + \frac{1}{L_a} e_a$$

$$T_m = k_i i_a$$

$$= J_m \frac{d^2\theta_m}{dt^2} + B_m \frac{d\theta_m}{dt} + T_L$$

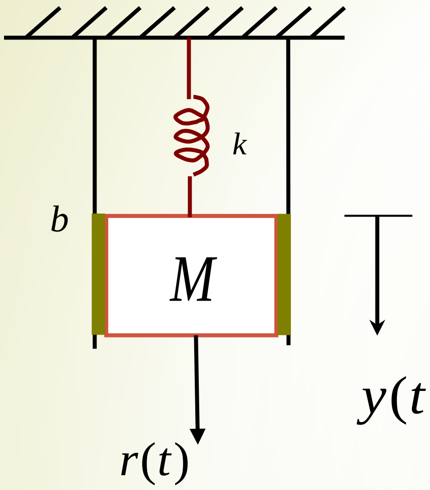
$$\frac{d\omega_m}{dt} = \frac{k_i}{J_m} i_a - \frac{B_m}{J_m}\frac{d\theta_m}{dt} - \frac{1}{J_m} T_L$$

$$\frac{d\theta_m}{dt} = \omega_m$$

$$\begin{bmatrix} \dot{i}_a \\ \dot{\omega}_m \\ \dot{\theta}_m \end{bmatrix} = \begin{bmatrix} -\frac{R_a}{L_a} & \frac{-K_b}{L_a} & 0 \\ \frac{K_i}{J_m} & \frac{-B_m}{J_m} & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \omega_m \\ \theta_m \end{bmatrix} + \begin{bmatrix} \frac{1}{L_a} & 0 \\ 0 & -\frac{1}{J_m} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_a \\ T_L \end{bmatrix}$$

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \theta_m(t) \\ \omega_m(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} i_a \\ \omega_m \\ \theta_m \end{bmatrix}$$

Example



$y(t),\ \dot{y}(t)$

r: *external force*

k: *spring*

b: *coefficient of viscous friction*

By Newton's Law

$$F = M\ddot{y}$$

$$r - ky - b\dot{y} = M\ddot{y}$$

$$\Rightarrow M\ddot{y} + b\dot{y} + ky = r$$

let $\boxed{x_1 = y, x_2 = \dot{y}}$

$$\Rightarrow \begin{cases} \dot{x}_1 = \dot{y} = x_2 \\ \dot{x}_2 = \ddot{y} = -\dfrac{b}{M}\dot{y} - \dfrac{k}{M}y + \dfrac{1}{M}r \end{cases}$$

$$= -\frac{b}{M}x_2 - \frac{k}{M}x_1 + \frac{1}{M}u \qquad (u = r)$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\dfrac{k}{M}x_1 - \dfrac{b}{M}x_2 + \dfrac{1}{M}u \end{cases}$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\dfrac{k}{M} & -\dfrac{b}{M} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix}}_{B} \cdot u \qquad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u$$

State Space Equation

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

For example :
$$\begin{cases} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\dfrac{k}{M} & -\dfrac{b}{M} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \dfrac{1}{M} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 0 \cdot u \end{cases}$$

Transfer Function

$$G(s) = \frac{Y(s)}{U(s)}$$

For example : $G(s) = \dfrac{b_1 s + b_0}{a_2 s^2 + a_1 s + a_0}$

Example: Transfer function of the Mass-damper-spring system

$$M\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = u(t)$$

$$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$$

$$Ms^2Y(s) + bsY(s) + kY(s) = U(s)$$

$$\frac{Y(s)}{U(s)} = G(s) = \frac{1}{Ms^2 + bs + k}$$

Example

MIMO system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}0 & 1 & 0\\ 0 & -4 & 3\\ -1 & -1 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}u_1\\ u_2\end{bmatrix}$$

$$\begin{bmatrix}y_1(t)\\ y_2(t)\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

$$(sI-A)^{-1}=\frac{adj(sI-A)}{\left|sI-A\right|}$$

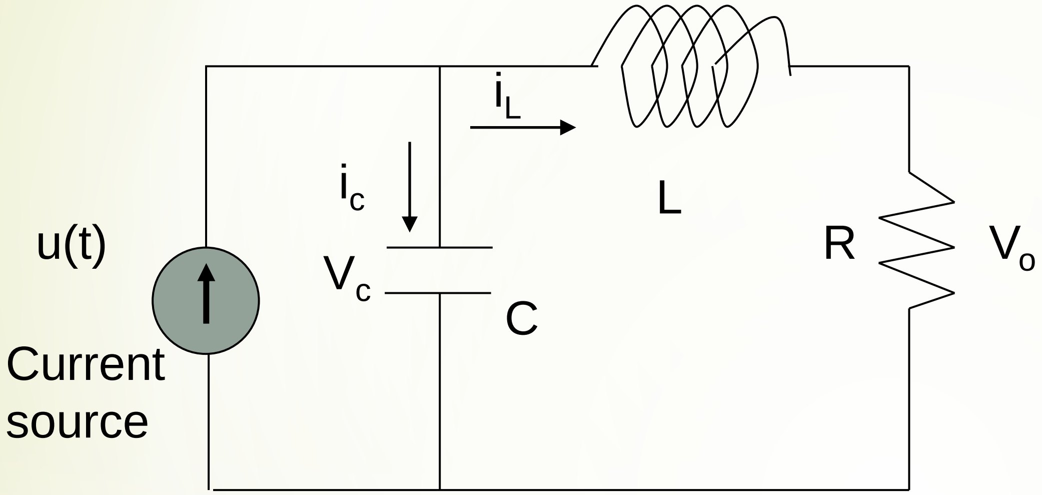
$$=\frac{1}{s(s+4)(s+2)+3+3s}\begin{bmatrix}s^2+6s+11 & s+2 & 3\\ -3 & s^2+2 & 3s\\ s+4 & -s-1 & s^2+4s\end{bmatrix}$$

$$G(s)=[C(sI-A)^{-1}B+D]$$

$$=\frac{1}{s^3+6s^2+11s+3}\begin{bmatrix}s+2 & 3\\ -(s+1) & s(s+4)\end{bmatrix}$$

Transfer function

As another example of the state variable characterization of a system, consider the RLC circuit shown in the following figure.



The state of this system can be described in terms of a set of variables $[x_1\ x_2]$, where x_1 is the capacitor voltage $v_c(t)$ and x_2 is equal to the inductor current $i_L(t)$. This choice of state variables is intuitively satisfactory because the stored energy of the network can be described in terms of these variables.

$$E_1 = \frac{1}{2} L i_L{}^2, \quad E_2 = \frac{1}{2C}\left(\int i_c dt\right)^2 = \frac{1}{2} C v_c^2$$

Therefore $x_1(t_0)$ and $x_2(t_0)$ represent the total initial energy of the network and thus the state of the system at $t=t_0$.

Utilizing Kirchhoff’s current low at the junction, we obtain a first order differential equation by describing the rate of change of capacitor voltage

$$i_c = C\frac{dv_c}{dt} = u(t) - i_L$$

Kirchhoff’s voltage low for the right-hand loop provides the equation describing the rate of change of inducator current as

$$L\frac{di_L}{dt} = -R\,i_L + v_c$$

The output of the system is represented by the linear algebraic equation

$$v_0 = R\,i_L(t)$$

We can write the equations as a set of two first order differential equations in terms of the state variables x_1 [$v_C(t)$] and x_2 [$i_L(t)$] as follows:

$$C\frac{dv_c}{dt} = u(t) - i_L \quad \Longrightarrow \quad \frac{dx_1}{dt} = -\frac{1}{C}x_2 + \frac{1}{C}u(t)$$

$$L\frac{di_L}{dt} = -R\,i_L + v_c \quad \Longrightarrow \quad \frac{dx_2}{dt} = \frac{1}{L}x_1 - \frac{R}{L}x_2$$

The output signal is then $y_1(t) = v_0(t) = R\,x_2$

Utilizing the first-order differential equations and the initial conditions of the network represented by [$x_1(t_0)$ $x_2(t_0)$], we can determine the system’s future and its output.

The state variables that describe a system are not a unique set, and several alternative sets of state variables can be chosen. For the RLC circuit, we might choose the set of state variables as the two voltages, $v_C(t)$ and $v_L(t)$.

In an actual system, there are several choices of a set of state variables that specify the *energy stored in a system* and therefore adequately describe the dynamics of the system.

The state variables of a system characterize the dynamic behavior of a system. The engineer’s interest is primarily in physical, where the variables are voltages, currents, velocities, positions, pressures, temperatures, and similar physical variables.

The State Differential Equation:

The state of a system is described by the set of first-order differential equations written in terms of the state variables [x_1 x_2 ... x_n]. These first-order differential equations can be written in general form as

$$
\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \ldots a_{1n}x_n + b_{11}u_1 + \cdots b_{1m}u_m \\
\dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \ldots a_{2n}x_n + b_{21}u_1 + \cdots b_{2m}u_m \\
\vdots \\
\dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \ldots a_{nn}x_n + b_{n1}u_1 + \cdots b_{nm}u_m
\end{aligned}
$$

Thus, this set of simultaneous differential equations can be written in matrix form as follows:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \cdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & \cdots & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}$$

n: number of state variables, m: number of inputs.

The column matrix consisting of the state variables is called the **state vector** and is written as

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

The vector of input signals is defined as u. Then the system can be represented by the compact notation of the state differential equation as

$$\dot{x} = Ax + Bu$$

This differential equation is also commonly called the state equation. The matrix **A** is an nxn square matrix, and **B** is an nxm matrix. The state differential equation relates the rate of change of the state of the system to the state of the system and the input signals. In general, the outputs of a linear system can be related to the state variables and the input signals by the output equation

$$y = Cx + Du$$

Where **y** is the set of output signals expressed in column vector form. The state-space representation (or state-variable representation) is comprised of the state variable differential equation and the output equation.

We can write the state variable differential equation for the RLC circuit as

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u(t)$$

and the output as

$$y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

The solution of the state differential equation can be obtained in a manner similar to the approach we utilize for solving a first order differential equation. Consider the first-order differential equation

$$\dot{x} = ax + bu$$

Where x(t) and u(t) are scalar functions of time. We expect an exponential solution of the form $e^{at.}$ Taking the Laplace transform of both sides, we have

$$s\,X(s) - x_0 = a\,X(s) + b\,U(s)$$

therefore,

$$X(s) = \frac{x(0)}{s-a} + \frac{b}{s-a}U(s)$$

The inverse Laplace transform of X(s) results in the solution

$$x(t) = e^{at}x(0) + \int_0^t e^{a(t-\tau)}b\,u(\tau)\,d\tau$$

We expect the solution of the state differential equation to be similar to x(t) and to be of differential form. The **matrix exponential function** is defined as

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots + \frac{A^kt^k}{k!} + \cdots$$

which converges for all finite t and any A. Then the solution of the state differential equation is found to be

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$X(s) = [sI - A]^{-1}x(0) + [sI - A]^{-1}BU(s)$$

where we note that $[sI\text{-}A]^{-1}=\phi(s)$, which is the Laplace transform of $\phi(t)=e^{At}$. The matrix exponential function $\phi(t)$ describes the unforced response of the system and is called the fundamental or state transition matrix.

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$

THE TRANSFER FUNCTION FROM THE STATE EQUATION

The transfer function of a single input-single output (SISO) system can be obtained from the state variable equations.

$$\dot{x} = Ax + Bu$$
$$y = Cx$$

where y is the single output and u is the single input. The Laplace transform of the equations

$$sX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s)$$

where B is an nx1 matrix, since u is a single input. We do not include initial conditions, since we seek the transfer function. Reordering the equation

$$[sI - A]X(s) = B\,U(s)$$

$$X(s) = \left[sI - A\right]^{-1} BU(s) = \phi(s)BU(s)$$

$$Y(s) = C\phi(s)BU(s)$$

Therefore, the transfer function G(s)=Y(s)/U(s) is

$$G(s) = C\phi(s)B$$

Example:

Determine the transfer function G(s)=Y(s)/U(s) for the RLC circuit as described by the state differential function

$$\dot{x} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix} u \quad , \quad y = \begin{bmatrix} 0 & R \end{bmatrix} x$$

$$[sI - A] = \begin{bmatrix} s & \frac{1}{C} \\ -\frac{1}{L} & s + \frac{R}{L} \end{bmatrix}$$

$$\phi(s) = [sI - A]^{-1} = \frac{1}{\Delta(s)} \begin{bmatrix} s + \frac{R}{L} & -\frac{1}{C} \\ \frac{1}{L} & s \end{bmatrix}$$

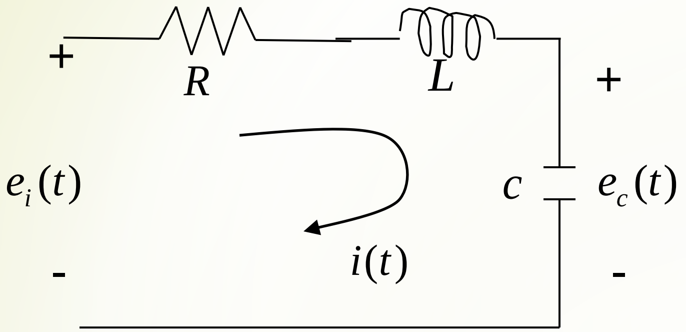
$$\Delta(s) = s^2 + \frac{R}{L}s + \frac{1}{LC}$$

Then the transfer function is

$$G(s) = \begin{bmatrix} 0 & R \end{bmatrix} \begin{bmatrix} \dfrac{s + \frac{R}{L}}{\Delta(s)} & -\dfrac{1}{C\Delta(s)} \\ \dfrac{1}{L\Delta(s)} & \dfrac{s}{\Delta(s)} \end{bmatrix} \begin{bmatrix} \frac{1}{C} \\ 0 \end{bmatrix}$$

$$G(s) = \frac{R/LC}{\Delta(s)} = \frac{R/LC}{s^2 + \frac{R}{L}s + \frac{1}{LC}}$$

Remark : the choice of states is not unique.



$$Ri(t) + L\frac{di(t)}{dt} + \frac{1}{c}\int_0^t i(t)dt = e_i(t)$$

let
$$x_1(t) = i(t)$$
$$x_2(t) = \int i(t)dt$$
$$y(t) = i(t)$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-\frac{R}{L} & -\frac{1}{LC}\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}e_i(t)$$

$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

exist a mapping

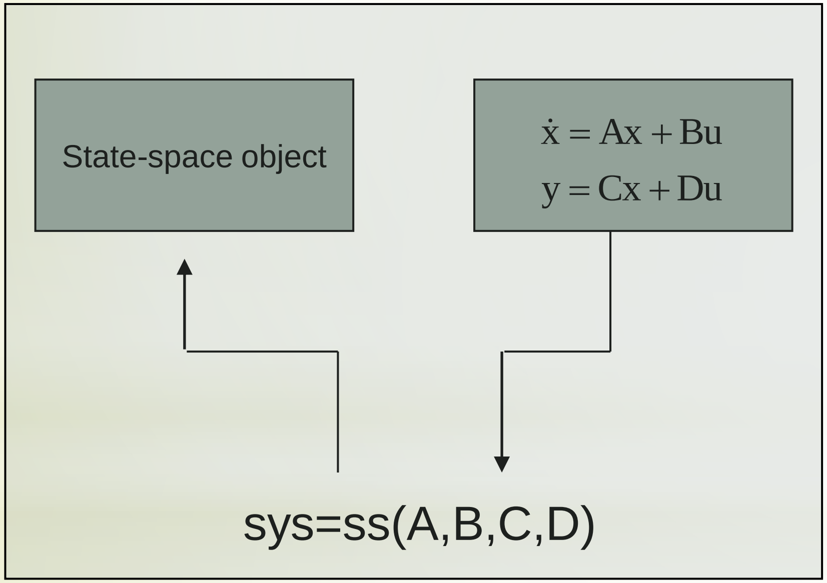
let
$$\hat{x}_1(t) = i(t)$$
$$\hat{x}_2(t) = e_c(t)$$
$$y(t) = i(t)$$

$$\begin{bmatrix}\dot{\hat{x}}_1\\ \dot{\hat{x}}_2\end{bmatrix} = \begin{bmatrix}-\frac{R}{L} & -\frac{R}{L}\\ \frac{1}{L} & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix} + \begin{bmatrix}\frac{1}{L}\\ 0\end{bmatrix}e_i(t)$$

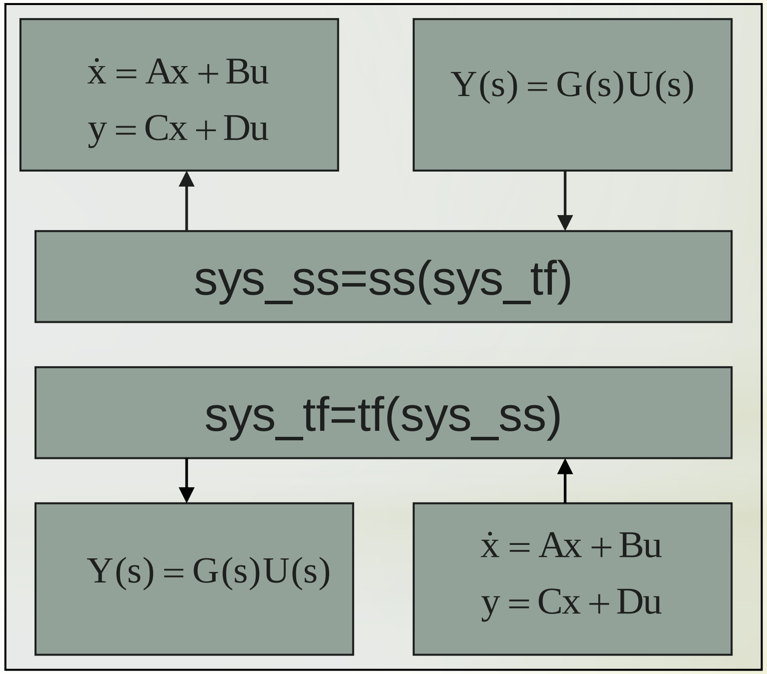
$$y(t) = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}\hat{x}_1\\ \hat{x}_2\end{bmatrix}$$

ANALYSIS OF STATE VARIABLE MODELS USING MATLAB

Given a transfer function, we can obtain an equivalent state-space representation and vice versa. The function **tf** can be used to convert a state-space representation to a transfer function representation; the function **ss** can be used to convert a transfer function representation to a state-space representation. The functions are shown in Figure 4, where sys_tf represents a transfer function model and sys_ss is a state space representation.



The **ss** function



Linear system model conversion

For instance, consider the third-order system

$$G(s)=\frac{Y(s)}{R(s)}=\frac{2s^{2}+8s+6}{s^{3}+8s^{2}+16s+6}$$

We can obtain a state-space representation using the **ss** function. The state-space representation of the system given by G(s) is

Matlab code

```
num=[2 8 6];den=[1 8 16 6];

sys_tf=tf(num,den)

sys_ss=ss(sys_tf)
```

Answer

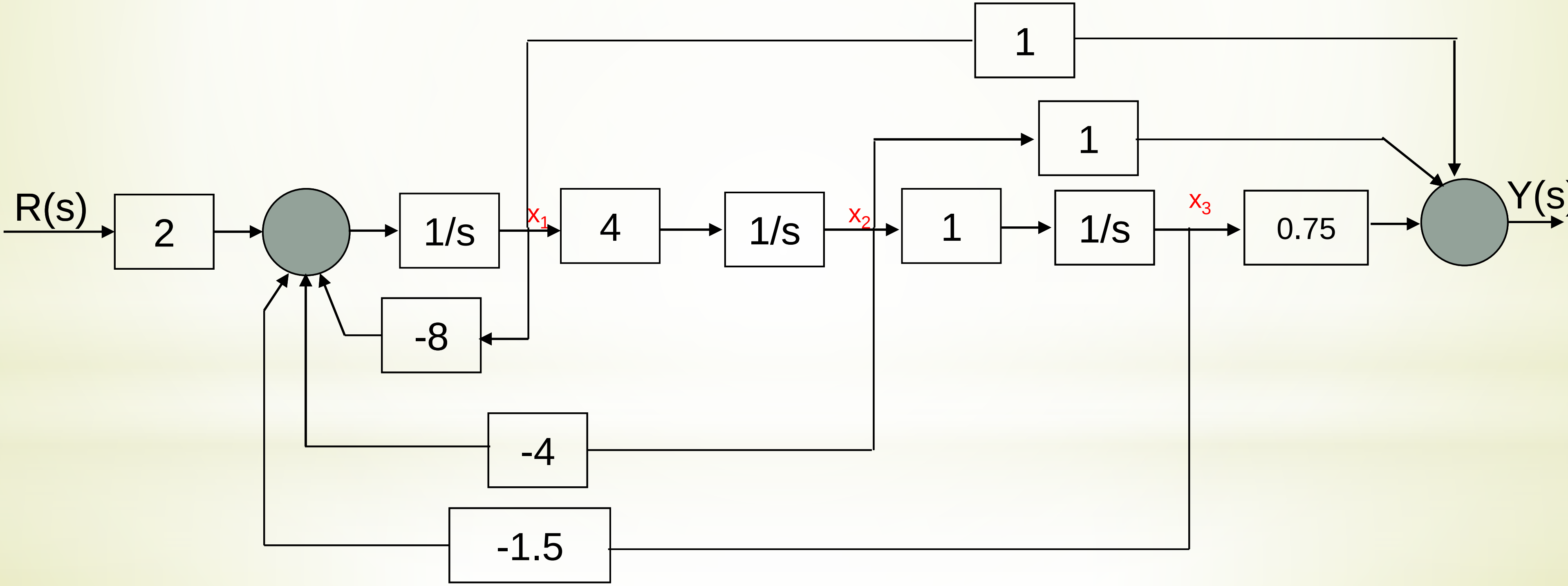
```
Transfer function:
  2 s^2 + 8 s + 6
----------------------
s^3 + 8 s^2 + 16 s + 6


a =
       x1    x2    x3
  x1   -8    -4  -1.5
  x2    4     0     0
  x3    0     1     0

b =
      u1
  x1   2
  x2   0
  x3   0

c =
       x1    x2    x3
  y1    1     1  0.75

d =
      u1
  y1   0

Continuous-time model.
```

$$A=\begin{bmatrix}-8 & -4 & -1.5\\ 4 & 0 & 0\\ 0 & 1 & 0\end{bmatrix},\ B=\begin{bmatrix}2\\ 0\\ 0\end{bmatrix}$$

$$C=\begin{bmatrix}1 & 1 & 0.75\end{bmatrix}\text{and } D=\begin{bmatrix}0\end{bmatrix}$$

$$A = \begin{bmatrix} -8 & -4 & -1.5 \\ 4 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 1 & 0.75 \end{bmatrix} \text{and } D = \begin{bmatrix} 0 \end{bmatrix}$$



Block diagram with x_1 defined as the leftmost state variable.

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$$

$$x(t) = \phi(t)\,x(0) + \int_0^t \phi(t-\tau)\,Bu(\tau)\,d\tau$$

We can use the function **expm** to compute the transition matrix for a given time. The **expm(A)** function computes the matrix exponential. By contrast the exp(A) function calculates e^{a}_{ij} for each of the elements a_{ij}ϵA.

For the RLC network, the state-space representation is given as:

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix} \text{and } D = \begin{bmatrix} 0 \end{bmatrix}$$

The initial conditions are $x_1(0)=x_2(0)=1$ and the input u(t)=0. At t=0.2, the state transition matrix is calculated as

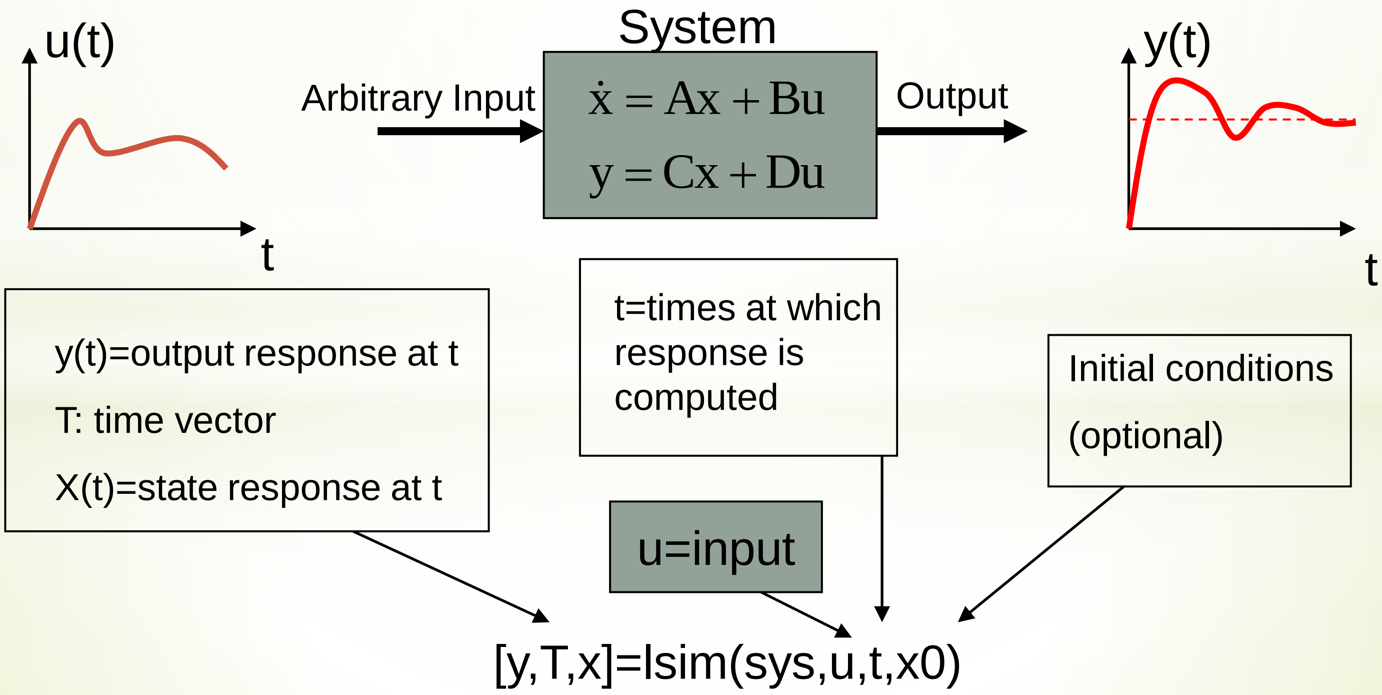
```
>>A=[0 -2;1 -3], dt=0.2; Phi=expm(A*dt)
```

```
Phi =

    0.9671   -0.2968
    0.1484    0.5219
```

The state at t=0.2 is predicted by the state transition method to be

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}_{t=0.2} = \begin{bmatrix}0.9671 & -0.2968\\0.1484 & 0.5219\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}_{t=0} = \begin{bmatrix}0.6703\\0.6703\end{bmatrix}$$

The time response of a system can also be obtained by using **lsim** function. The **lsim** function can accept as input nonzero initial conditions as well as an input function. Using **lsim** function, we can calculate the response for the RLC network as shown below.



y(t)=output response at t

T: time vector

X(t)=state response at t

t=times at which response is computed

Initial conditions

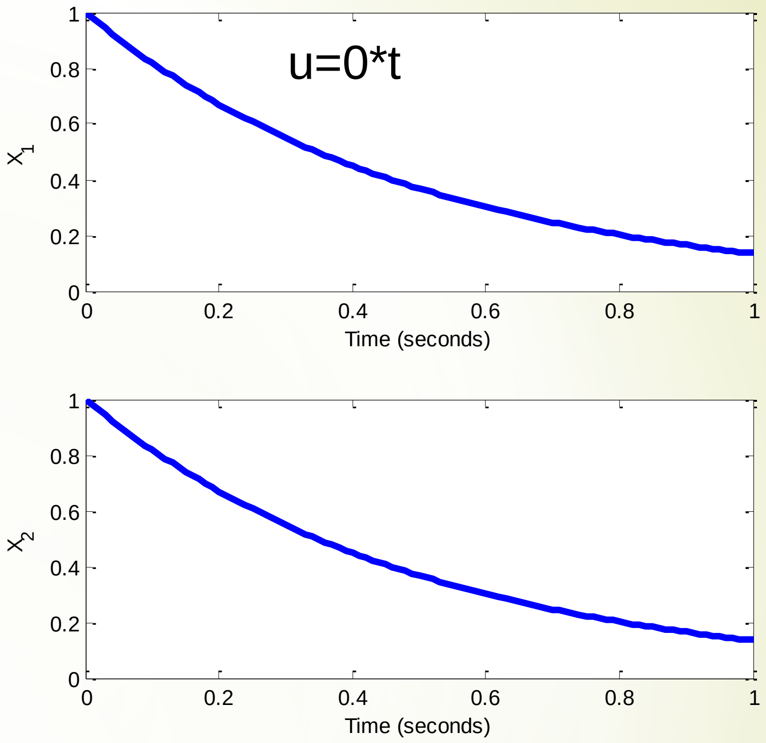
(optional)

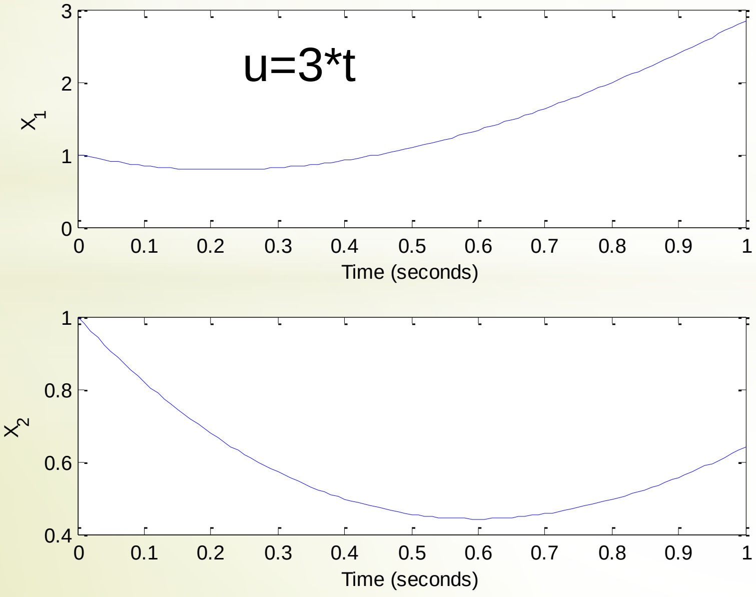
u=input

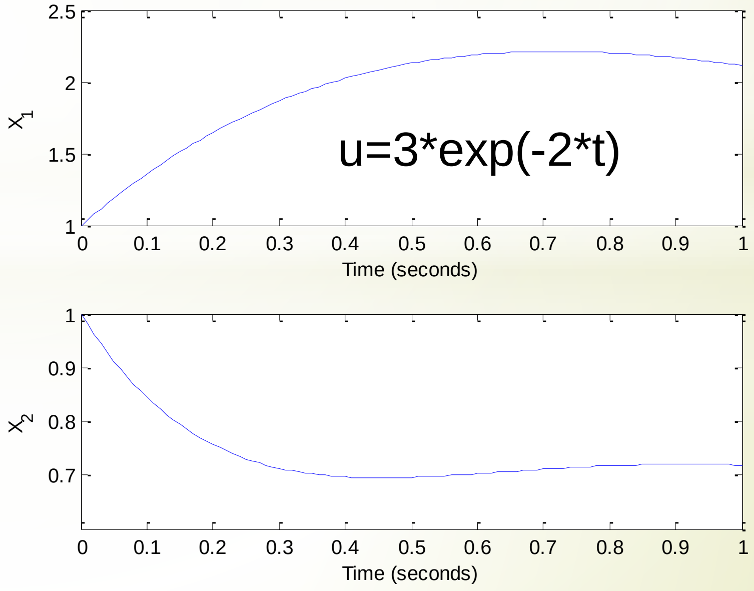
[y,T,x]=lsim(sys,u,t,x0)

Matlab code

```
clc;clear
A=[0 -2;1 -3];B=[2;0];C=[1 0];D=[0];
sys=ss(A,B,C,D)   %state-space model
x0=[1 1]; %initial conditions
t=[0:0.01:1];
u=0*t;  %zero input
[y,T,x]=lsim(sys,u,t,x0);
subplot(211),plot(T,x(:,1))
xlabel('Time (seconds)'),ylabel('X_1')
subplot(212),plot(T,x(:,2))
xlabel('Time (seconds)'),ylabel('X_2')
```







With Our Best Wishes

Automatic Control (1)

Course Staff

Associate Prof. Dr . Mohamed Ahmed Ebrahim

Thank You

For Your Attention

Mohamed Ahmed Ebrahim